

OPTIMIZATION OF FILTER BANKS BASED ON PROPERTIES OF THE INPUT SIGNAL

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Abstract

Filter banks and wavelets have found applications in signal compression, noise removal, and in many other signal processing contexts. In this tutorial we review a number of recent results on the optimization of filter banks based on the knowledge of the input. The main emphasis will be the minimization of error due to subband quantization, and its connection to principal component reconstruction. Both uniform and nonuniform filter banks are considered.

1. INTRODUCTION

Figure 1(a) shows a general M channel maximally decimated filter bank [32], [36], where the decimation ratios n_k are integers satisfying $\sum_{k=0}^{M-1} 1/n_k = 1$. For the uniform case where $n_k = M$ for all k , we have the polyphase representation of Fig. 1(b). The general expression for the reconstructed signal is

$$\hat{x}(n) = \sum_{k=0}^{M-1} \sum_{m=-\infty}^{\infty} y_k(m) f_k(n - n_k m)$$

The system is said to be a perfect reconstruction (PR) system if $\hat{x}(n) = x(n)$ in absence of subband quantization. This is also called the **biorthogonality** condition. For such a system, the preceding equation therefore is a representation of $x(n)$ as a linear combination of the sequences $\eta_{k,m}(n) = f_k(n - n_k m)$. The decimated subband signals $y_k(m)$ are the transform coefficients, and $\{\eta_{k,m}(n)\}$ constitutes a filter-bank basis for $x(n)$. An **orthonormal** filter bank is one for which this basis is orthonormal, i.e.,

$$\sum_n \eta_{k,m}(n) \eta_{\ell,i}(n) = \delta(k - \ell) \delta(m - i)$$

For the case of uniform filter banks orthonormality is equivalent to the unitary property of $\mathbf{E}(e^{j\omega})$, and such filter banks are also called **paraunitary** filter banks. The filter bank decomposition should be compared to the wavelet decomposition of continuous time signals given by

$$x(t) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{km} 2^{k/2} \psi(2^k t - m)$$

where the basis functions $\{2^{k/2} \psi(2^k t - m)\}$ are derived from a mother wavelet $\psi(t)$ by dilation and shifting. Notice in particular that the filter bank provides a general time-frequency representation for $x(n)$. Depending on the choice

of decimators $\{n_k\}$ and the filters $\{H_k\}$, a wide range of time-frequency representations are possible, leading to several applications. Examples include selective quantization and bit allocation (data compression), thresholding (denoising a signal), progressive or principal-component reconstruction, etc.

Wavelets and filter banks have been used extensively for image coding. A mathematical theory was developed in [6] wherein several error measures were analyzed and it was demonstrated that the use of L_1 -**norm** for the error is usually more in tune with the properties of the human visual system. The authors in [6] also show that the smoothness of an image, measured in accordance with membership in certain function spaces (Besov spaces) has a direct bearing on the quality of the compressed image for a given bit rate. More recently the **zero-tree coder**, originally proposed by Shapiro [23] and significantly improved by Said and Pearlman [20], has become the state of the art for compression of natural images. It has been shown by Gonzalez and Akansu [9] that the simpler form of **zero-zone quantizers** together with multiplierless QMF banks often result in systems that are quite comparable to the embedded zero tree coder. Since a state-of-the-art review has recently been presented by Pearlman and Said [20], we will not elaborate further on this.

In this paper we review a number of recent results on the optimization of filter banks based on the knowledge of the input. The main emphasis will be the minimization of error due to subband quantization, and its connection to principal component reconstruction.

2. OPTIMAL BIT ALLOCATION

In data compression applications the subband signals are quantized using an optimal bit allocation scheme. This problem has been considered by several authors in the last two decades [22], [24]. Some of these assume exponential **quantizer functions (QF)**, while some others have the milder requirement that the quantizer functions be convex. A common assumption in early work was that all quantizers have similar characteristics, that is, identical QF. Moreover the number of bits were often not restricted to be positive integers. Using marginal analysis techniques, Fox presented an algorithm to perform bit allocation under the positive integer constraint, and the convexity assumption on quantizers was later relaxed by Trushkin (see [24] for references). Remarkable generalizations of these “early” techniques were proposed by Westernik, et al [37] and Shoham and Gersho [24]. These authors recognized that in a truly optimal scheme the quantizer function should be allowed to be fairly arbitrary, especially at low rates. The quantizer function should in fact be “derived on line” depending on the data and the bit rate region.

Shoham and Gersho relax the convexity assumption; in

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fact the allocation works for any real, measured, quantizer characteristics. The bit-rate constrained optimization is turned into an unconstrained family of problems which generate a family of solutions parameterized by a Lagrange multiplier λ . Shoham and Gersho also develop a sound theoretical framework which leads to efficient procedures for “sweeping through” λ to arrive at the correct bit rate with minimum distortion. Under high bit rate assumptions, however, a simple closed form expression can be derived for the optimal b_i [32,8].

By extending the work of Shoham and Gersho, Ramchandran and Vetterli [21] developed an elegant method for the generation of the appropriate tree structure for subband coding of signals. This results in a system which is optimal in the rate distortion sense, for a given bit rate. The tree structured filter bank also defines a more general time-frequency representation, namely the **wavelet packet**, of which the uniform (Fourier style) representation and the dyadic (wavelet style) representation are mere special cases. The data-dependent tree generation allows one to take into account the nonstationary nature of real signals. It offers more efficient, signal adaptive, time-frequency localization.

3. FILTER BANK OPTIMIZATION

There is substantial amount of literature on the optimization of the filter coefficients in a filter bank to achieve a certain goal. In these cases, the filter bank or time-frequency tiling scheme is typically fixed (e.g., a uniform filter bank, or a dyadic tree structured filter bank) and the filter bank coefficients optimized.

3.1. Optimizing the Filter Bank for Coding

The *coding gain* of a subband coder is the improvement in the reconstruction mse, over the mse that would result from direct uniform quantization (for the same average bit rate). For the uniform orthonormal filter bank, and under high bit rate assumptions (Sec. 4), the coding gain is the ratio of the arithmetic mean to geometric mean of subband variances $\sigma_{y_i}^2$. Akansu and Liu showed [3] how to numerically optimize the filter coefficients for a composite objective function which includes the coding gain as one term. Uzun and Haddad [31] modelled the subband quantization error and optimized the filters for the two channel filter bank assuming a WSS input with known psd, and assuming that the quantizers are pdf optimized. More recent theoretical results on coding gain optimization [30, 35] will be reviewed in Sec. 4–6. This includes the development of optimality conditions for the uniform orthonormal subband coder. The optimal biorthogonal filter bank problem (which is still open) is considered in Sec. 5. Several observations pertaining to the optimality of nonuniform (e.g., tree structured) filter banks will be presented in Sec. 6.

3.2. Advantage of Biorthogonal Filter Banks

The advantage of biorthogonal filter banks over orthonormal ones has been established in a number of papers. In the two channel case, orthonormal filter banks cannot have linear phase analysis filters (unless the filters are two-tap filters) which seems to be desirable for image coding applications. Biorthogonal filter banks have therefore been quite successful in image coding [4]. More recently [16] Malvar has developed clever special cases of the general biorthogonal system, which take care of blocking and ringing artifacts in image coding very effectively. On the the-

oretical side, Djokovic and Vaidyanathan established a result in [8]. They constrained a filter bank to be of the form in Fig. 2 and solved for the best combination of the **prefilter part** and **orthonormal part** under high bit rate assumption. The result is that the prefilter should be the optimal half-whitening filter for the given input $x(n)$ (i.e., $P(e^{j\omega}) = 1/S_{xx}^{1/4}(e^{j\omega})$) and that the orthonormal part should be the optimal orthonormal filter bank for the input $x(n)$. Thus the two optimizations can be done separately. Aase and Ramstad solved a slightly different problem in [2]. They assumed a stationary AR(1) model and that the filters are ideal non overlapping filters. Under this assumption they showed that the analysis filter bank can be decomposed into an orthonormal part and a subband halfwhitening part. This can effectively be redrawn as in Fig. 2. The superiority of this system over orthonormal filter banks is then established with image coding experiments. However it has not been established in the literature that the system of Fig. 2 is as good as the most general biorthogonal system. This was claimed implicitly in [1], but an error in the proof there makes it inconclusive.

3.3. Non Biorthogonal Filter Banks

The output $\hat{x}(n)$ of a biorthogonal filter bank can be written as

$$\hat{x}(n) = x(n) + e(n)$$

where $x(n)$ is the input and $e(n)$ is the quantization noise filtered through the synthesis bank. We could reduce the effect of this noise by using a Wiener filter at the output of the synthesis bank (so the overall system is not biorthogonal). This would be a periodically time varying filter because the reconstruction noise $e(n)$ is in general cyclo stationary (assuming quantizers produce WSS noise). Expressions for this filter under various statistical assumptions have been presented in [33]. The idea of using a post-filter to reduce the noise has been proposed by a number of authors independently [11,5,19,25].

4. THEORY OF OPTIMAL SUBBAND CODERS

It turns out to be very difficult to develop a theory for optimization of filter coefficients in filter banks without making **high bit rate** assumptions on quantizers. This assumption allows the quantizers to be represented with white uncorrelated additive noise sources with noise variances

$$\sigma_{q_i}^2 = c\sigma_{y_i}^2 2^{-2b_i} \quad (1)$$

where $\sigma_{y_i}^2$ is the variance of the input to the i th quantizer (which is assigned b_i bits). Such assumptions, while unrealistic in traditional image coding, appears to be well justified for special applications such as the compression of multispectral imagery and to some extent in audio coding.

4.1. Problem Formulation

In Fig. 3, $\mathbf{x}(n)$ is the blocked version of the scalar process $x(n)$ which is input to the subband coder. We assume $\mathbf{x}(n)$ is zero-mean wide sense stationary (WSS). The quantizer error vector is $\mathbf{q}(n) = \hat{\mathbf{y}}(n) - \mathbf{y}(n)$ and the reconstruction error vector is $\mathbf{e}(n) = \hat{\mathbf{x}}(n) - \mathbf{x}(n)$. The noise sources $q_i(n)$ are assumed to be zero-mean, uncorrelated, and white, with variances given by Eq. (1). Assuming **optimal bit allocation** with fixed bit rate $b = \sum_i b_i/M$, the m.s.e is

$$\mathcal{E} \triangleq E[\mathbf{e}^\dagger(n)\mathbf{e}(n)] = cM2^{-2b}\phi^{1/M} \quad (2)$$

$$\phi \triangleq \prod_{i=0}^{M-1} \int_0^{2\pi} \left(\mathbf{E} \mathbf{S}_{xx} \mathbf{E}^\dagger \right)_{ii} \frac{d\omega}{2\pi} \int_0^{2\pi} \left(\mathbf{R}^\dagger \mathbf{R} \right)_{ii} \frac{d\omega}{2\pi} \quad (3)$$

where \mathbf{S}_{xx} is the psd matrix of the vector $\mathbf{x}(n)$. The aim therefore is to choose $\mathbf{E}(e^{j\omega})$ and $\mathbf{R}(e^{j\omega})$ subject to the biorthogonality constraint $\mathbf{R}(e^{j\omega})\mathbf{E}(e^{j\omega}) = \mathbf{I}$, such that the **objective function** ϕ is minimized.

4.2. Orthonormal Case

For the orthonormal case $\mathbf{R}^\dagger(e^{j\omega})\mathbf{R}(e^{j\omega}) = \mathbf{I}$, so that

$$\phi = \prod_{i=0}^{M-1} \int_0^{2\pi} \left(\mathbf{E} \mathbf{S}_{xx} \mathbf{E}^\dagger \right)_{ii} \frac{d\omega}{2\pi} = \prod_{i=0}^{M-1} \sigma_{y_i}^2 \quad (4)$$

where $\sigma_{y_i}^2$ is the variance of $y_i(n)$. For this to be minimized, it is necessary [34] that the *decimated subband random processes* be uncorrelated, that is, $E[y_i(n)y_k^*(m)] = 0$, for $i \neq k$, and for all n, m . Equivalently, the psd matrix of $\mathbf{y}(n)$ must be diagonal:

$$\mathbf{S}_{yy}(e^{j\omega}) = \text{diag} \{S_0(e^{j\omega}), S_1(e^{j\omega}), \dots, S_{M-1}(e^{j\omega})\}$$

where $S_i(e^{j\omega})$ is the psd of $y_i(n)$. This condition is also called **total decorrelation** of subbands. For optimality in the orthonormal case a second property called **spectral majorization** is also necessary, namely,

$$S_0(e^{j\omega}) \geq S_1(e^{j\omega}) \geq \dots \geq S_{M-1}(e^{j\omega}), \quad \text{for all } \omega,$$

assuming appropriate ordering of subbands. Neither total decorrelation nor spectral majorization is sufficient for optimality, but if these are together satisfied by a given filter bank, we can show that ϕ is indeed minimized [34]. Thus total decorrelation and spectral majorization are together **necessary and sufficient**.

4.3. Principal Component Filter Banks

Consider the class \mathcal{K}_c of all M -band, uniform, orthonormal subband coders, subject to some additional constraints \mathcal{C} . The constraints, for example, could be that the filters be FIR or IIR, with order $\leq N$. Or the constraint could be that the filters be derived from a single prototype by cosine modulation. Assume the filters are always numbered such that the output variances are in decreasing order, that is, $\sigma_{y_0}^2 \geq \sigma_{y_1}^2 \dots$. A particular member of the class \mathcal{K}_c is said to be a **principal component filter bank** for this class if its subband variances $\{\hat{\sigma}_{y_k}^2\}$ satisfy

$$\sum_{k=0}^L \hat{\sigma}_{y_k}^2 \geq \sum_{k=0}^L \sigma_{y_k}^2$$

where $\sigma_{y_k}^2$ are the subband variances for any other member in the class. That is, all the partial sums of variances must be at least as large as the partial sums for any other class member. This property is useful in progressive transmission where we wish to transmit dominant subbands according to order of importance. Assuming there are no "additional constraints \mathcal{C} " we know that spectral majorization and total decorrelation are necessary and sufficient for mmse property. It can be shown that this solution also has the principal component property. For nonuniform filter banks, and for filter banks with other constraints (e.g., FIR, cosine modulation, etc.) there is no simple correspondence between mmse filter banks and principal component filter banks. See Sec. 6.

4.4. Energy Compaction Filters

A filter $H_0(e^{j\omega})$ is said to be an optimal compaction(M) filter for a WSS input $x(n)$ with psd $S_{xx}(e^{j\omega})$ if the output variance

$$\int_0^{2\pi} |H_0(e^{j\omega})|^2 S_{xx}(e^{j\omega}) d\omega / 2\pi$$

is maximized by designing $H_0(e^{j\omega})$ subject to the condition that $|H_0(e^{j\omega})|^2$ be a **Nyquist(M)** filter, i.e., a filter with impulse response $g(n)$ satisfying

$$g(Mn) = \delta(n)$$

The Nyquist(M) condition is used because it is naturally satisfied by every analysis filter in any orthonormal filter bank. With no order-constraints imposed on the filters, techniques for designing such filters have been outlined in [34].

Total decorrelation and spectral majorization can together be satisfied by designing the analysis filters to be optimal compaction filters for appropriate sections of the input psd $S_{xx}(e^{j\omega})$. Details can be found in several places, e.g., [34] or [35]. This is why energy compaction filters are important in subband coding. Several properties of optimal compaction filters can be found in [34].

Optimal compaction filters also arise in the design of principal component filter banks. Indeed, the role of energy compaction has been sufficiently emphasized in many related problems. For example, the success of the zero-tree wavelet coder [23] stems partly from the energy compaction property of wavelet transforms. The success of **wavelet denoising** methods [7] is also due to efficient compaction.

4.5. Optimal FIR Filter Banks

If the analysis and synthesis filters are confined to be FIR, then for arbitrary M very little can be said about the conditions for optimality in the mmse sense. For example, even in the uniform decimation case, we cannot say that the principal component solution also solves the mmse problem with quantizers. In fact, it has been shown in [14] that for the FIR case *there may not exist a principal component filter bank*. In particular the solution which maximizes the coding gain may not be such that the filter $H_0(z)$ is a compaction filter.

However, several useful suboptimal methods for coding gain optimization have in the past been reported for the FIR case [3], [31]. More recently, Moulin and Mihcak [18] have shown that we can obtain very good coding gain in an FIR orthonormal filter bank by designing the first filter $H_0(z)$ to be a compaction filter and designing the remaining $M-1$ filters such that the system $\{H_k(z)\}$ satisfies orthonormality. This is done by first synthesizing a cascaded lattice structure for $H_0(z)$ and then inserting a constant unitary matrix \mathbf{U} in front which defines the remaining filters. Notice that the optimal compaction solution specifies $|H_0(e^{j\omega})|^2$ but not the spectral factor $H_0(z)$. It has been shown in [14] that the exact choice of the spectral factor affects the coding gain, though the numerical differences are not very significant in practical examples.

FIR compaction filter design. Several methods have been reported in the literature for the design of FIR compaction filters. Moulin, et al. have used linear programming to obtain good FIR designs. A comprehensive summary of many methods can be found in [13], along with the introduction of a method called the window method. This method leads to very fast designs, and has the advantage of

simplicity at a slight cost of optimality. A technique based on some deep state space theoretical results (the Kalman-Yakubovic lemma) has been developed in [29], which allows truly optimal designs and has several advantages over other methods. Finally, examples of very efficient IIR compaction filters can be found in [28].

5. THE BIORTHOGONAL CASE

Let $\mathbf{E}(e^{j\omega}) = \mathbf{E}_{opt}(e^{j\omega})$ be an optimum biorthogonal solution (i.e., the one minimizing ϕ). Given any arbitrary paraunitary $\mathbf{U}(e^{j\omega})$, we can always write $\mathbf{E}_{opt}(e^{j\omega}) = \mathbf{G}(e^{j\omega})\mathbf{U}(e^{j\omega})$. Thus the polyphase part of the optimal biorthogonal system can be redrawn as in Fig. 4(a). Since $\mathbf{U}(e^{j\omega})$ is unitary, the errors $\mathbf{e}(n) = \hat{\mathbf{x}}(n) - \mathbf{x}(n)$ and $\mathbf{e}_w(n) = \hat{\mathbf{w}}(n) - \mathbf{w}(n)$ have equal m.s. values

$$E[\mathbf{e}_w^\dagger(n)\mathbf{e}_w(n)] = E[\mathbf{e}^\dagger(n)\mathbf{e}(n)].$$

Thus, though the choice of $\mathbf{U}(e^{j\omega})$ will certainly affect the reconstruction error for $\mathbf{w}(n)$, this error will always be equal to the error for $\mathbf{x}(n)$. It follows that $\mathbf{G}(e^{j\omega})$ is optimal for its input $\mathbf{w}(n)$, i.e., it minimizes the m.s. value of $\mathbf{e}_w(n)$. This reasoning holds for any paraunitary $\mathbf{U}(e^{j\omega})$. For example $\mathbf{U}(e^{j\omega})$ could be the optimal orthonormal system for $\mathbf{x}(n)$. Thus, we can always **decouple** the design of the optimal biorthogonal $\mathbf{E}(e^{j\omega})$ into two steps: (a) first design the optimal orthonormal system $\mathbf{U}(e^{j\omega})$ which produces an output $\mathbf{w}(n)$ satisfying *total decorrelation* and *spectral majorization*, and (b) design the best biorthogonal $\mathbf{G}(e^{j\omega})$ for $\mathbf{w}(n)$.

Case of diagonal \mathbf{G} . Suppose the paraunitary matrix $\mathbf{U}(e^{j\omega})$ performs total decorrelation and spectral majorization. In seeking the best biorthogonal system, is there still a loss of generality in restricting $\mathbf{G}(e^{j\omega})$ to be a diagonal matrix? Let us first find out how best we can do if we constrain $\mathbf{G}(e^{j\omega})$ to be diagonal. Denote the diagonal elements as $\lambda_i(e^{j\omega})$. Let $S_i(e^{j\omega})$ be the psd of the i th component $w_i(n)$ of $\mathbf{w}(n)$. We can write the m.s. value of $\mathbf{e}_w(n)$ as $cM2^{-2b}\phi_w^{1/M}$ where

$$\begin{aligned} \phi_w &= \prod_{i=0}^{M-1} \int_0^{2\pi} |\lambda_i|^2 S_i \frac{d\omega}{2\pi} \int_0^{2\pi} \frac{1}{|\lambda_i|^2} \frac{d\omega}{2\pi} \\ &\geq \left(\prod_{i=0}^{M-1} \int_0^{2\pi} \sqrt{S_i} \frac{d\omega}{2\pi} \right)^2 \end{aligned}$$

using Schwartz inequality. This holds with equality if we let $\lambda_i(e^{j\omega}) = 1/S_i^{1/4}(e^{j\omega})$, which is a **half-whitening filter**. If we assume that $\mathbf{U}(e^{j\omega})$ has performed total decorrelation and spectral majorization, then $S_i(e^{j\omega})$ are the majorized eigenvalues $\eta_i(e^{j\omega})$ of $\mathbf{S}_{xx}(e^{j\omega})$. So we have:

Lemma 1. In Fig. 4(b) suppose $\mathbf{U}(e^{j\omega})$ is the optimum orthonormal solution for $\mathbf{x}(n)$, and $\lambda_i(e^{j\omega})$ are the optimal (half-whitening) filters for their respective inputs. Then the objective function ϕ has the value $\phi_\eta \triangleq \left(\prod_{i=0}^{M-1} \int_0^{2\pi} \sqrt{\eta_i(e^{j\omega})} \frac{d\omega}{2\pi} \right)^2$ \diamond

It is clear that the special form of biorthogonal system shown in Fig. 4(b) achieves $\phi = \phi_\eta$. An **open problem** here is to prove (or disprove, say, by a counter example) that the optimal biorthogonal system satisfies $\phi_{opt} = \phi_\eta$

or, equivalently, that

$$\phi \geq \phi_\eta \quad (5)$$

for any biorthogonal filter bank. If this is true, the optimal biorthogonal system can be represented as in Fig. 4(b).

5.1. The PreFilteredOrthonormal Filter Bank

If we convert the polyphase system in Fig. 4(b) into the normal form like Fig. 1(a), we obtain the structure shown in Fig. 2. The details of this derivation can be found in [35]. Here the filters $P_i(e^{j\omega})$ are the analysis filters of the orthonormal SBC corresponding to the polyphase matrix $\mathbf{U}(e^{j\omega})$. The half-whitening filters have been effectively moved past the decimators, and $P_i(e^{j\omega})$ and lumped into a single prefilter $P(e^{j\omega})$. As shown in [35] this has been possible partly with the help of noble identities [32] and partly because the optimal orthonormal filters $P_i(e^{j\omega})$ can be assumed, without loss of generality, to be nonoverlapping ideal filters. Since $\lambda_i(e^{j\omega})$ are the half-whitening solutions for the disjoint portions of the input psd created by the analysis system $\{P_i(e^{j\omega})\}$, it follows that we can take $P(e^{j\omega})$ to be the half-whitening filter for the scalar subband coder input $x(n)$, namely $P(e^{j\omega}) = 1/S_{xx}^{1/4}(e^{j\omega})$. The optimal prefiltered orthonormal filter bank has the m.s.e

$$\mathcal{E} = cM2^{-2b}\phi_\eta^{1/M} = cM2^{-2b} \left(\prod_{i=0}^{M-1} \int_0^{2\pi} \sqrt{\eta_i} \frac{d\omega}{2\pi} \right)^{2/M}$$

where $\eta_i(e^{j\omega})$ are the power spectra of the decorrelated majorized subbands $y_i(n)$.

In an earlier work [8], the above combination of $P(e^{j\omega})$ and $\{P_i(e^{j\omega})\}$ was shown to be optimum among the class of prefiltered orthonormal filter banks (i.e., filter banks restricted to the form of Fig. 2). At the time of this writing we do not know if the most general optimal biorthogonal system can always be represented in this form; but at least for a restricted class of power spectra this can be shown to be the case [34].

5.2. Performance Bounds

Recall that the reconstruction error under optimal bit allocation is given by $\mathcal{E} = cM2^{-2b}\phi^{1/M}$. For any choice of $\mathbf{E}(e^{j\omega})$ we can show [35]

$$\phi \geq \prod_{i=0}^{M-1} \left| \int_0^{2\pi} \left(\mathbf{E} \mathbf{S}_{xx}^{1/2} \mathbf{E}^{-1} \right)_{ii} \frac{d\omega}{2\pi} \right|^2$$

According to the singular value decomposition theorem, the nonsingular matrix $\mathbf{E}(e^{j\omega})$ can be expressed as $\mathbf{E}(e^{j\omega}) = \mathbf{V}(e^{j\omega})\mathbf{\Lambda}(e^{j\omega})\mathbf{U}(e^{j\omega})$ where $\mathbf{U}(e^{j\omega})$ and $\mathbf{V}(e^{j\omega})$ are unitary and $\mathbf{\Lambda}(e^{j\omega})$ is a diagonal matrix with positive diagonal elements $\lambda_i(e^{j\omega})$. By expressing the preceding bound in terms of these matrices, some conclusions can be drawn [35]:

1. If we impose the condition $\mathbf{V}(e^{j\omega}) = \mathbf{I}$ and then optimize $\mathbf{U}(e^{j\omega})$ and $\mathbf{\Lambda}(e^{j\omega})$, then the resulting ϕ satisfies $\phi \geq \phi_\eta$, where ϕ_η is defined in Lemma 1.
2. Instead of imposing the condition $\mathbf{V}(e^{j\omega}) = \mathbf{I}$, suppose we impose the condition that $\mathbf{U}(e^{j\omega})$ be chosen to perform total decorrelation of its output. If we now try to optimize $\mathbf{\Lambda}(e^{j\omega})$ and $\mathbf{V}(e^{j\omega})$, then the resulting ϕ again satisfies $\phi \geq \phi_\eta$.

3. In any case the solution $\phi = \phi_\eta$ can be achieved by the following choice of matrices: (a) set $\mathbf{V} = \mathbf{I}$, (b) take $\mathbf{U}(e^{j\omega})$ to be the optimal paraunitary matrix (the one that performs total decorrelation and spectral majorization), and (c) take the diagonal elements of $\mathbf{\Lambda}(e^{j\omega})$ to be the half-whitening filters for the outputs of $\mathbf{U}(e^{j\omega})$ (Fig. 4(b)).

See [35] for other bounds based on $\det[\mathbf{S}_{xx}(e^{j\omega})]$.

6. NONUNIFORM FILTER BANKS

The nonuniform filter bank of Fig. 1(a) is orthonormal if the analysis filters satisfy $H_k(e^{j\omega})H_m^*(e^{j\omega})\big|_{\downarrow g_{km}} = \delta(k - m)$ where $g_{km} = \gcd\{n_k, n_m\}$. The coding gain of the nonuniform M -band orthonormal filter bank is [8]

$$G_{SBC}(M) = \frac{\sigma_x^2}{\prod_{i=0}^{M-1} (\sigma_{y_i}^2)^{1/n_i}} \quad (6)$$

Consider the special case of a dyadic tree structured filter bank (Fig. 5), which is equivalent to a M -channel nonuniform FB. Assuming each two channel filter bank is orthonormal, we have an M -band nonuniform orthonormal filter bank. The coding gain of the nonuniform system is

$$G_{SBC}(M) = G_1 G_2^{1/2} G_3^{1/4} \dots \quad (7)$$

where G_m is the coding gain at level m . Thus the extra benefit offered by the m th split decays exponentially with m . Another consequence of (7) is that, if the tree structure maximizes the coding gain for a given input then the *right-flushed subtrees indicated in Fig. 6 are optimal for their respective inputs*. However, since these inputs depend on the filters preceding them, this observation is not directly useful to identify the optimal filter bank.

6.1. Coding Gain and Compaction Gain

Consider the input power spectrum shown in Fig. 7(a). Assuming that the process is Gaussian, the rate-distortion theoretic upper bound on the coding gain [12], given by $G_{up} = \sigma_x^2 / \exp \int_0^{2\pi} \ln S_{xx}(e^{j\omega}) d\omega / 2\pi$, has the value

$$G_{up} = \frac{2 + c + d}{4(cd)^{1/4}} \quad (8)$$

Suppose we use the orthonormal tree structured system shown in Fig. 5, with two levels, and choose the filter responses as in Fig. 7(b) and 7(c). The coding gains of the individual levels are

$$G_1 = \frac{2 + c + d}{4\sqrt{\frac{c+d}{2}}}, \quad G_2 = \frac{c + d}{2\sqrt{cd}}$$

The total coding gain $G_1 G_2^{1/2}$ is therefore as in Eq. (8), showing that the tree achieves the upper bound on the coding gain. This means in particular that the choice of filters shown in Fig. 7 results in an optimal orthonormal two-level dyadic tree. Next, Fig. 7(d) shows the effective filter $H_{10}(z)H_{20}(z^2)$ of the top channel (which has effective decimator $\downarrow 4$) of the two-level filter bank. Since $c < 1$, this is clearly not an optimum compaction(4) filter for the input psd. This example shows that, even though the coding gain is optimized, the top filter $H_{10}(z)H_{20}(z^2)$ is not an optimum compaction filter for the input $x(n)$.

6.2. Principal Components and Compaction

As shown above, unlike in uniform filter banks, the coding gain and compaction gain are not directly related in the case of nonuniform orthonormal filter banks. However, there is a simple direct relation between optimal compaction and the principal component property even in the nonuniform case.

In our discussion we find it convenient to change the normalization convention for the filters. Thus consider the example of a 3-level tree of the form Fig. 5. Assuming that the filter bank is orthonormal we have $\sigma_{y_0}^2 + \sigma_{y_1}^2 + 2\sigma_{y_2}^2 + 4\sigma_{y_3}^2 = 8\sigma_x^2$. We would like to rescale the analysis filters such that under the new scaling convention $\sigma_{y_0}^2 + \sigma_{y_1}^2 + \sigma_{y_2}^2 + \sigma_{y_3}^2 = \sigma_x^2$, or more generally, $\sum_{k=0}^{M-1} \sigma_{y_k}^2 = \sigma_x^2$. This is accomplished by dividing each two-channel analysis filter $H_{km}(z)$ by $\sqrt{2}$ and multiplying the corresponding synthesis filter by $\sqrt{2}$. Under the new scaling convention, the signals s_k in Fig. 5 have variances given by

$$\sigma_{s_L}^2 = \sum_{k=0}^L \sigma_{y_k}^2$$

If the tree structured filter bank has the principal component property for the given input $x(n)$, then the partial sum $\sigma_{s_L}^2$ is maximized for each value of L in $0 \leq L \leq M-1$. That is, the left-flushed subtrees indicated in Fig. 8 should be such that their top filters

$$H_{10}(z)H_{20}(z^2) \dots H_{i+1,0}(z^{2^i})$$

are optimal compaction(2^{i+1}) filters for the primary input $x(n)$. For example, $H_{10}(z)$ should be the optimum compaction(2) filter for $x(n)$, whereas $H_{10}(z)H_{20}(z^2)$ must be the optimum compaction(4) filter for the same $x(n)$ and so forth. To design the principal component system, consider the example of a three-level tree. First design an optimum compaction(8) filter for the input $x(n)$. This filter can always be implemented in the multirate-cascade form shown in Fig. 9 where each $H_{k0}(z)$ is an optimal compaction(2) filter for its input. Since each $H_{k0}(z)$ is such that $|H_{k0}(e^{j\omega})|^2$ is Nyquist(2), we can always define a filter $H_{k1}(z)$ such that the pair $\{H_{k0}(z), H_{k1}(z)\}$ is a two-channel orthonormal filter bank. Since each $H_{k0}(z)$ is an optimum compaction filter for its input, the pair $\{H_{k0}(z), H_{k1}(z)\}$ maximizes the coding gain for its input. In this way, the complete tree structure is defined (by Fig. 5) and satisfies the principal component property. The coding gain of this principal component filter bank, however, is not necessarily maximized.

7. CONCLUDING REMARKS

Several levels of sophistication of the subband quantizers are available, the simplest version being the uniform quantizer. The use of differential coding in the lowpass subband results in significant compression. The use of **pdf optimized**, or ‘‘Lloyd-Max’’ quantizers has been studied by a number of authors as well. With such quantizers, the error is orthogonal to the quantized signal regardless of the bit rate. However, a ‘‘gain plus additive noise model’’ is more appropriate than a simple additive noise model [15, 38]. The most recent and promising kind of subband quantization incorporates the zero-tree coder [23] which has become the state of the art in image coding.

A problem closely related to filter bank optimization is **wavelet basis optimization**. Tewfik, et al. have con-

sidered the difficult problem of finding an optimal compactly supported wavelet representation for a given finite length signal, e.g., a speech segment (with signal duration typically \gg mother wavelet duration). The optimality is in the sense of minimizing the error when the wavelet representation is truncated at a desired scale. While this is a difficult problem in general, the authors are able to provide upper bounds on the truncation errors, and then minimize the bounds numerically. The resulting approximations turn out to be very useful if not optimal. In a later paper Gopinath et al. have generalized this signal dependent optimization in a number of ways; e.g., they considered M -adic wavelets (generated from M -band filter banks) and proposed different norms for optimization.

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